Linear Algebra II

07/04/2021, Wednesday, 18:45 – 21:45 (deadline for handing in: 22:05)

- This Take-Home Exam is 'open-book', which means that the book as well as lecture notes may be used as a reference. The exam contains 5 problems.
- For handing in the exam, the use of electronic devices is of course allowed. The student is fully responsible for handing in his/her complete work before the deadline. You are asked to upload your answers as a **pdf-file**.
- Every student must upload the signed declaration before the start of the exam. An exam will not be graded in case the signed declaration has not been uploaded. After grading, short discussions with (a selection of) students will be held to check for possible fraud.
- Write your name and student number on each page!

1 $(9 + 9 = 18$ pts) **Least squares approximation**

Consider the real inner product space $C[-1, 1]$ with inner product $\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx$. Let $S \subseteq C[-1, 1]$ be the subspace spanned by the functions 1, x and x^2.

(a) Determine an orthonormal basis of S.

(b) Compute the orthogonal projection of the function x^3 onto the subspace S.

2 $(5 + 5 + 8 = 18$ pts) **Eigenvalues**

Let A be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and Jordan form J. Let $q(s)$ be a polynomial with complex coefficients.

(a) Let k be a nonnegative integer. Argue that J^k is upper triangular, and determine its diagonal elements

(b) Determine the eigenvalues of the matrix $q(J)$.

(c) Show that $q(A)$ and $q(J)$ have the same eigenvalues.

3 $(4+5+5+4 = 18$ pts) **Positive definite matrices**

Let A be a real $n \times n$ matrix. Consider the linear matrix inequality

$$A^\top X + XA < 0$$

in the unknown symmetric matrix $X \in \mathbb{R}^{n\times n}$.

(a) Show that if the inequality has a positive definite solution solution X, then every eigenvalue λ of A satisfies $\mathrm{Re}(\lambda) < 0$.

(b) Assume all eigenvalues λ of A satisfy $\mathrm{Re}(\lambda) < 0$. This implies that $\lim_{t\to\infty} e^{A^\top t}e^{At} = 0$ and the integral

$$X := a \int_0^\infty e^{A^\top t}e^{At}dt \quad (\star)$$

exists. Now, let $a > 0$ be given and consider the linear equation

$$A^\top X + XA = -aI$$

Show that the integral $(\star)$ is a solution of this linear equation.

(c) Prove that X given by $(\star)$ is positive definite.

(d) Prove that the linear matrix inequality $A^\top X + XA < 0$ has a positive definite solution X if and only if all eigenvalues λ of A satisfy $\mathrm{Re}(\lambda) < 0$.

4 $(9+9 = 18$ pts) **Singular value decomposition**

Consider the vector $u_1 \in \mathbb{R}^3$ defined as

$$u_1 := \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

(a) Compute vectors $u_2, u_3 \in \mathbb{R}^3$ such that the matrix $U := (u_1\ u_2\ u_3)$ is an orthogonal matrix.

Consider now the matrix

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

(b) Compute a singular value decomposition of A.

5 $(4+3+4+4+3 = 18 \text{ pts})$ **Jordan Form**

Let $A \in \mathbb{C}^{4\times 4}$ be defined as

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$$

Here $a, b \in \mathbb{C}$ are given constants.

(a) Determine for all values of a and b the eigenvalues of A and their corresponding algebraic multiplicities.

(b) Give necessary and sufficient conditions on a and b so that A is in Jordan normal form.

(c) Assume $a = b$. Determine the geometric multiplicity of each eigenvalue of A.

(d) Assume that $a \neq b$. Determine the Jordan normal form of A.

(e) Assume $a \neq b$. Determine the minimal polynomial of A

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10 pts free